# Limit theorems for functionals of stationary Gaussian fields

Leonardo Maini (Università degli Studi di Milano-Bicocca)

Stochastics and Geometry

Banff, 12/09/2024

Based on joint papers with Nikolai Leonenko (Cardiff University), Ivan Nourdin (University of Luxembourg) and Francesca Pistolato (University of Luxembourg).

# Our problem

- $B = (B_x)_{x \in \mathbb{R}^d}$ **stationary Gaussian field**, $B_x \sim N(0,1)$.
- $B$ has **covariance function** $C : \mathbb{R}^d \to \mathbb{R}$

$$C(x-y) := \mathrm{Cov}(B_x, B_y)\,, \qquad x, y \in \mathbb{R}^d\,.$$

- Fix $D \subseteq \mathbb{R}^d$ compact, $\varphi : \mathbb{R} \to \mathbb{R}$. Consider the **functional**

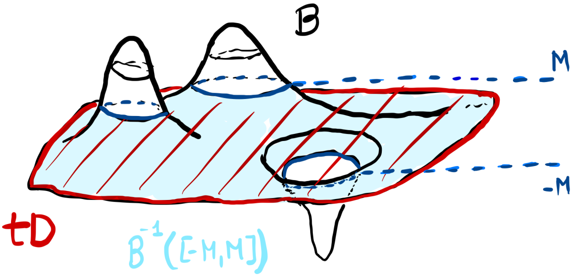
$$Y(t) := \int_{tD} \varphi(B_x)\, dx$$

**Problem:** $\tilde{Y}(t) := \dfrac{Y(t) - \mathbb{E}[Y(t)]}{\sqrt{\mathrm{Var}(Y(t))}} \xrightarrow{d} ?$ as $t \to \infty$.

# Geometric functionals

- We focus on $\varphi = \mathbf{1}_{[-M,M]}$

$$Y(t) = \int_{tD} \varphi(B_x)\, dx = \mathrm{Vol}\left(\{|B_x| \le M\} \cap tD\right)$$



# Short memory and CLTs

**Definition**

$(A_x)_{x\in\mathbb{R}^d}$ stationary, with covariance $K(x) = \mathrm{Cov}(A_0, A_x)$ has short (long) memory if $K \in L^1(\mathbb{R}^d)$ ($K \notin L^1(\mathbb{R}^d)$).

$$Y(t) := \int_{tD} A_x \, dx = \int_{tD} \mathbf{1}_{[-M,M]}(B_x) \, dx$$

$(B_x)_{x\in\mathbb{R}^d}$ with covariance function $C$. Then

$\mathbf{1}_{[-M,M]}(B_x)$ has short (long) memory if $C \in L^2$ ($C \notin L^2$).

# Short/long memory and limit in distribution

$$Y(t) := \int_{tD} \mathbf{1}_{[-M,M]}(B_x)\, dx$$

**Theorem (Breuer-Major, *J. Multiv. Analysis, 1983*)**

$C \in L^2$ **(short memory)** $\implies \tilde{Y}(t) \xrightarrow{d} N(0,1)$.

**Theorem (Dobrushin-Major (*PTRF, 1979*); Taqqu (*PTRF*, 1975))**

$C(x) \asymp \frac{1}{\|x\|^\beta}$ , $\beta \in (0, \frac{d}{2})$ **(long memory)** $\implies \tilde{Y}(t) \overset{d}{\not\to} N(0,1)$.

**Warning!** Intuition on short/long memory can be **misleading**.

# CLTs in the case of long memory: motivating examples

- Berry's random wave model: $B$ with covariance function

$$C(x) = \int_{S^{d-1}} e^{i\langle x,\theta\rangle} \frac{d\theta}{\omega_{d-1}} \approx \frac{\cos(\|x\| - c_d)}{\|x\|^{\frac{d}{2}-\frac{1}{2}}},$$

- $B$ with separable covariance function $C : \mathbb{R}^{d_1+d_2} \to \mathbb{R}$

$$C(x_1, x_2) = C_1(x_1)C_2(x_2) \approx \frac{1}{\|x_1\|^{\frac{d_1}{2}+\epsilon}} \frac{1}{\|x_2\|^{\frac{d_2}{2}-\epsilon}}, \qquad x_i \in \mathbb{R}^{d_i}.$$

E.g.: rectangular increments of a fractional Brownian sheet.

# CLT in the case of long memory

**Theorem (M., Nourdin, *Ann. Probab.*, 2024)**

*If* $d \geq 2$, $C(x) = \rho(\|x\|)$, $C(x) = \int_{\mathbb{R}^d} e^{i\langle x, \lambda \rangle} G(d\lambda)$, *then*

$$\int_{\mathbb{R}^d} \|\lambda\|^{-\frac{d}{2}} G(d\lambda) < \infty \implies \tilde{Y}(t) \xrightarrow{d} N(0,1)$$

Berry's random wave model: $B$ with covariance function

$$C(x) = \int_{S^{d-1}} e^{i\langle x, \theta \rangle} \frac{d\theta}{\omega_{d-1}} \approx \frac{\cos(\|x\| - c_d)}{\|x\|^{\frac{d}{2} - \frac{1}{2}}},$$

# CLT for $p$-domain functionals

$$Y(t_1, \ldots, t_p) = \int_{t_1 D_1 \times \cdots \times t_p D_p} \mathbf{1}_{[-M,M]}(B_{(x_1,\ldots,x_p)})\, dx_1 \ldots dx_p \,.$$

**Main motivation: spatio-temporal functionals.**

**Theorem (Leonenko, M., Nourdin, Pistolato, (*EJP, 2024+.*))**

$C(x_1, \ldots, x_p) = C_1(x_1) \ldots C_p(x_p)$ , $C \in \bigcup_M L^M$ *then*

$$C_i \in L^2(\mathbb{R}^{d_i}) \quad \textit{for some } i \quad \Longrightarrow \quad \widetilde{Y}(t_1, \ldots, t_p) \to N(0,1).$$

$$C \notin L^2, \quad C_i \in L^2(\mathbb{R}^{d_i}) \quad \Longrightarrow \quad \tilde{Y}(t_1, \ldots, t_p) \to N(0,1)\,.$$

# Sketch of the proof

- Step 1: $C_i$ is a covariance function (of a Gaussian field $B^{(i)}$).

$$Y_i(t_i) = \int_{t_i D_i} \mathbf{1}_{[-M,M]}\left(B^{(i)}_{x_i}\right)\, dx_i$$

- Step 2: $\mathbf{1}_{[-M,M]}(x)$ can be replaced by $H_2(x) = x^2 - 1$.
- Step 3: use 4th moment theorems (Nualart, Peccati, *Ann. Probab.*, 2005); (Nourdin, Peccati, *PTRF*, 2009);...
- Step 4: prove separability for the fourth cumulants.

Thank you!